Some insights from total collapse

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Abstract

We discuss the Sundman-Weierstrass theorem of total collapse in its historical context. This remarkable and relatively simple result, a type of stability criterion, is at the crossroads of some interesting developments in the gravitational Newtonian N-body problem. We use it as motivation to explore the connections to such important concepts as integrability, singularities and typicality in order to gain some insight on the transition from a predominantly quantitative to a novel qualitative approach to dynamical problems that took place at the end of the 19th century.

Keywords: Celestial Mechanics; N-body problem; total collapse, singularities.

I Introduction

Celestial mechanics is one of the treasures of physics. With a rich and fascinating history, [1] it had a pivotal role in the very creation of modern science. After all, it was Hooke's question on the two-body problem and Halley's encouragement (and financial resources) that eventually led Newton to publish the *Principia*, a watershed. [2] Conversely, celestial mechanics was the testing ground *par excellence* for the new mechanics, and its triumphs in explaining a wealth of phenomena were decisive in the acceptance of the Newtonian synthesis and the "clockwork universe".

From the beginning, special attention was paid to the N-body problem, the study of the motion of N massive bodies under mutual gravitational forces, taken as a reliable model of the the solar system. The list of scientists that worked on it, starting with Newton himself, makes a veritable hall of fame of mathematics and physics; also, a host of concepts and methods were created which are now part of the vast heritage of modern mathematical-physics: the calculus, complex variables, the theory of errors and statistics, differential equations, perturbation theory, potential theory, numerical methods, analytical mechanics, just to cite a few.

It is true that around the second-half of the 19th century, the mainstream of physics was less interested in what might have seemed a noble but old-fashioned

subject, and attention turned to the exciting new areas of thermodynamics and electromagnetism (and later, relativity, quantum theory and astrophysics). However, the field continued to be pursued by first-rate mathematicians and astronomers, who didn't lose sight of its enduring relevance. Henri Poincaré, one of the masters of the field, put it with utmost clarity when he observed that, besides the compilation of ephemeris, "the ultimate goal of celestial mechanics is to resolve the great problem of determining if Newton's laws alone explain all astronomical phenomena". [3] In this regard, recall that one of the first breakthroughs of Einstein's general relativity theory was an explanation of the discrepancy in the observed precession of Mercury's orbit, a long-standing open problem in classical celestial mechanics.

The arrival of the space-age (and race) and the advances in computer technology in the middle of the 20th century, sparked a renewed interest in celestial mechanics, in connection to problems of spacecraft navigation and solar systems dynamics. New exciting ideas appeared, particularly from the American and soviet schools: new approaches to perturbation theory (as in KAM theory), the challenges of chaotic behavior (whose existence was first glimpsed in Poincaré's work on the three-body problem), new tools from nonlinear dynamics, all leading to a rethinking of the implications to the long-time behavior of the solar system (the old stability problem).

This trend continues today, with an intense cross-fertilization between the highly abstract tools from nonlinear dynamical systems and sophisticated computer simulations, spurred by applications to astrodynamics. [4] Recent highlights are the unraveling of the interplanetary superhighway and the discovery of new solutions of the three-body problem. [5] These developments should be a sobering alert against a certain "anti-classical" bias of modern physics education, which may impart to students the impression that classical physics is completed (or stagnant), as if the only interesting and important questions lie in the quantum realm.

In this paper we present the Sundman-Weierstrass theorem of total collapse in its historical context. This relatively simple result, which is a kind of stability condition for the N-body problem, deserves to be better known. Its surprisingly simple proof, using a little calculus and linear algebra, can be profitably presented in a general mechanics course. Besides, it can be used as motivation to address many issues of historical and conceptual interest. Our main aim is to stimulate the reader's curiosity to explore this vast field and to experiment a bit of what had been aptly described as "the sheer joy of celestial mechanics". [6]

The paper is structured as follows. Section II describes the mathematical formulation of the N-body problem and the question of its integrability. In section III we see how celestial mechanics reached a crisis near the end of the 19th century, which revolved around the changing notion of a solution of the N-body problem. Section IV deals with the intriguing issue of singularities and the need to circumvent them in the quest for exact solutions for the three-body problem; this leads directly to the total collapse theorem which is also proved there for the general case of the N bodies. We conclude by arguing that the events described were part of a major conceptual change that took place in dynamics, leading from a predominantly quantitative to

a new qualitative approach, the impacts of which are still being assimilated.

II The N-body problem and its integrability

A fundamental problem of celestial mechanics is *the N-body problem*, which consists in applying Newton's laws of mechanics and his law of universal gravitation to an isolated system of N point masses moving in three-dimensional space. With respect to an inertial reference frame and disregarding the influence of other bodies, we have for the position $\mathbf{r}_j$ of the jth body,

$$m_j\ddot{\mathbf{r}}_j = \nabla_{\mathbf{r}_j} U(\mathbf{x}) = \Big(\frac{\partial U}{\partial x_{j1}}(\mathbf{x}), \frac{\partial U}{\partial x_{j2}}(\mathbf{x}), \frac{\partial U}{\partial x_{j3}}(\mathbf{x})\Big), \tag{1}$$

plus the initial conditions: at the initial time t_0, we are given the positions $\mathbf{r}_j(t_0) \neq \mathbf{r}_j(t_0)$, $i \neq j$, and velocities e $\mathbf{v}_j(t_0) = \dot{\mathbf{r}}_j(t_0)$. Here, $\mathbf{x} = (\mathbf{r}_1, \ldots, \mathbf{r}_N)$ is the system's configuration and U is the gravitational potential energy (using the sign convention used in celestial mechanics):

$$U(\mathbf{x}) \equiv \sum_{1\leq j<k\leq N} \frac{Gm_i m_j}{\|\mathbf{r}_k - \mathbf{r}_j\|}, \tag{2}$$

where $\|\cdot\|$ is the usual euclidean distance. Despite being an idealization (almost a caricature) of real gravitational systems, this model is nonetheless very successful and still in current use. [7]

As is well known, Newton's equations have an equivalent Hamiltonian version, namely, for $j = 1, \ldots, N$:

$$\begin{cases} \dot{\mathbf{r}}_j = \nabla_{\mathbf{p}_j} H \\ \dot{\mathbf{p}}_j = -\nabla_{\mathbf{r}_j} H \end{cases} \tag{3}$$

where $\mathbf{p}_j = m_j\mathbf{v}_j$, $H = T - U$ is the Hamiltonian function and with given initial condition $(\mathbf{x}(t_0), \mathbf{p}(t_0))$ belonging to the system's *phase-space* $(\mathbb{R}^{3N} - \Delta) \times \mathbb{R}^{3N}$. [8]

Mathematically, (3) is a system of $6N$ non-linear first-order differential equations; together with the initial data it defines an initial value problem. Faced with such a problem, the first task of a modern mathematician would be to prove the existence and uniqueness of solutions. To a physicist or engineer, however, "this is like leaving a restaurant without having eaten anything, but having paid the cover charge for the privilege of reading the menu". [9] Here we have a classical instance of the clash of styles of physicists and mathematicians, an endless source of witticisms. [10]

Though there is a genuine difference of concerns, it is more fruitful to see these approaches as complementary. So, if a physicist thinks his model gives a reasonable description of the phenomenon at hand, he is confident (maybe more than the mathematician) the formalism should be sound, and proceeds to get the consequences of it. On the other hand, an existence and uniqueness theorem (EUT) is not a mere

academic exercise; [11] besides vindicating the soundness of the formalism, it is the basis of the influential principle of Laplacian (or "mechanical") determinism: given the initial conditions, the future and the past are completely determined. Interestingly, the EUT is usually a purely local result in that it only asserts the existence of solutions for a small time interval. What kind of "catastrophe" could prevent the existence of global solutions? We'll come back to this in section IV.

Meanwhile, we observe that traditionally the main interest regarding the N-body problem was in its *integration*, that is, in finding its *solution* giving the positions of the bodies as "explicit" functions of time. It turns out that integrability is a thorny issue; to begin with, what is meant by such vague terms as "explicit function", "closed formula" or "analytic expression"?

Though there are different notions of integrability available, historically it usually referred to "integration by quadratures" (also known as "reduction"): a solution is to be found by performing a finite number of algebraic operations, integrations and inversions, over a class of known "elementary functions". Key to the method is finding so called "first integrals" (or constants of motion), i.e., functions of positions and momenta which are constant along the solution. As the level set of such a function (corresponding to given initial data) defines a hyper-surface in phase-space to which the solution curve belongs, the dimension of the problem is thereby reduced by one. Thus, if enough independent first integrals can be found the problem eventually becomes one-dimensional, and could be solved by "simple" integration.

The N-body problem has ten classical independent first integrals, namely: the total energy and the components of, respectively, the total linear momentum, the total angular momentum and the center-of-mass. [12] These correspond to the conservation laws associated to the Galilean invariance of classical mechanics. [13] So, in principle, the system could be reduced to a $(6N-11)$-dimensional one.

The two-body problem is integrable by quadratures as $6N-11=1$. However, already for the three-body problem the classical integrals are not enough, reduction leading to a 7-dimensional system. [14] This led to a conundrum clearly summarized by Wintner: [15]

> "When John and James Bernoulli, Clairaut, D'Alambert, D. Bernoulli, and Lambert, Euler and, finally, Lagrange applied the principles of Newton to the various problems of celestial and terrestrial mechanics, they had to face an awkward situation. For, on the one hand, it was almost axiomatic that a dynamical problem is 'solved' only if it is reduced to quadratures (and successive differentiations and eliminations); while on the other hand, the most urgent problems were almost never reducible by quadratures."

Of course, people had used other means, if not to exactly solve, at least to extract useful information from the N-body problem; for instance, looking for special solutions (say, with some symmetry) or using perturbation and numerical methods. However, and probably under the spell of the success with the two-body problem,

there was a widespread conviction that sooner or later the "geometers" would find the missing integrals, which would lay bare the secrets of the three-body problem. [16] The search for new integrals began to look like a kind of holy grail saga.

III The Oscar prize

Though integration by quadratures was made mathematically rigorous by Sophus Lie, [17] there are some crucial caveats regarding its implementation. Even if one is lucky to identify enough first integrals, it doesn't necessarily follow that a "closed form" expression for the solution can be obtained. To begin with, this would require the integrals to be not too complicated functions of the dynamical variables, allowing explicit isolation of some coordinates at various stages; and even the minimalist choice of algebraic functions gives no guarantee. Moreover, the operations involved in the method (function inversion, integration, etc), can easily take one out of the set of "elementary functions".

By the way, it should be emphasized that even for the two-body problem there is no such simple "closed formula" for the position of the bodies: it entails solving Kepler's equation, an implicit transcendental equation usually solved by a series expansion involving special functions. [18] What is remarkable about the two-body problem is that one can classify the possible orbits: they are the conic sections. This is in stark contrast to the three-body problem whose possible motions are *much* more complex and poorly understood.

Near the end of the 19th century there was a growing suspicion that maybe there aren't enough integrals for the three-body problem. Then, the hope to find "simple" first integrals was dashed by an impossibility (or "no-go") theorem due to Heinrich Bruns (1887) which states that no additional independent integrals exist which are algebraic functions of position and velocities (in Cartesian coordinates). [19]

Contrary to what is frequently asserted, this doesn't mean that the three-body problem is unsolvable, just that the *method* of quadratures won't work for it. By that time, solutions expressed as an infinite *series* was increasingly gaining acceptance (after all, perturbative expansions had been used for quite some time) and there was great anticipation for the general *exact* series solution to the N-body problem. This optimism is quite evident in the announcement of the famous prize problem sponsored by King Oscar II of Sweden in 1885: [20]

> "A system being given of a number whatever of particles attracting one another mutually according to Newton's law, it is proposed, on the assumption that there never takes place an impact of two particles to expand the coordinates of each particle in a series proceeding according to some known functions of time and converging uniformly for any space of time.
>
> It seems that this problem, the solution of which will considerably enlarge our knowledge with regard to the system of the universe, might be solved

by the analytical resources at our present disposition;..."

Moreover, it was widely believed that the old problem of the stability of the solar system (which goes back to Newton) would then be settled once and for all.

As is well known, Poincaré won the prize, not by actually solving the problem, but due to the many outstanding new ideas and methods he devised to attack it. One surprising result concerned the Lindstedt series, a perturbation expansion widely used in celestial mechanics: he showed that while using a few terms of the series worked well, the complete series was usually divergent!

So, near the end of the 19th century a disturbing collection of negative results began to accumulate, that brought celestial mechanics to a sort of crisis centered on the vexing question of what should count as a *bona fide* solution of the three-body problem. Some 30 years later, the Finnish astronomer Karl F. Sundman finally solved the prize problem exactly as required but, alas, it didn't fulfill the expectations.

IV Singularities and the total collapse theorem

Sundman was after global solutions, i.e., valid for all time, as required by the prize problem. Thus, the first difficulty he had to overcome was that of avoiding initial conditions that lead to a *singularity.* At this point, it is important to recall that the EUT is a purely *local* result: it only guarantees the existence of solutions for a small time interval around the initial time t_0. Of course, one could then try to enlarge this existence interval by patching up: pick a time instant t_1 near the boundary of that interval and use $(\mathbf{x}(t_1), \mathbf{p}(t_1))$ as new initial conditions; apply the EUT to enlarge the interval, and so on. If this can be continued forever, we eventually obtain a global or *regular* solution, which is defined for all times past and future. If, however, there happens to exist a (future or past) time instant t^* beyond which the solution *cannot* be extended, then we have what is called a *singular* solution. As we take regular solutions for granted, it is natural to ask: what is the nature of singularities? And how "rare" are they (in some sense)?

The first question above was dealt with by Paul Painlevé in 1897. As could be suspected, singularities might appear if particles get too close to each other so that the potential diverges and the equations of motion break down. In fact, Painlevé proved that a solution $(\mathbf{x}(t), \mathbf{p}(t))$ has a singularity at time t^* if, and only if, $\rho(t) \equiv \min_{j\neq k} r_{jk}(t)$ goes to zero as t approaches t^* (where $r_{jk} = \|\mathbf{r}_k - \mathbf{r}_j\|$). [21]

A *collision* is an obvious type of singularity, whose avoidance is clearly required in the prize problem. In a collision at time t^*, at least one pair of particles occupy the same position. However, it is conceivable that $\rho(t)$ could tend to zero without the occurrence of a collision: as t tends to t^*, the system could experience a sequence of "close approaches", with particles almost colliding, but then separating, to come even closer later, etc, in a wild oscillatory motion. This non-collisional type of singularity, dubbed a *pseudocollision*, is quite weird: in 1908 the Swedish astronomer

Hugo von Zeipel proved that it necessarily follows that some particles of the system go off to infinity in finite time. [22] Painlevé proved that pseudocollisions do not occur in the three-body problem; unable to extend the proof he conjectured that they can appear for $N \geq 4$. [23]

Back to the three-body problem, and building on these results, Sundman first realized that *binary* collisions are only apparent singularities: they can be "regularized" by a suitable re-parametrization so that the motion is continued beyond them much as if the bodies bounced each other elastically. [24] Unfortunately this trick doesn't work for *ternary* collisions [25] and therefore he looked for a criterion to avoid them.

Now, a ternary collision in the three-body problem is an instance of a *total collapse*: all particles collide at the same time at the same position. Sundman's criterion for avoiding these "catastrophes" is the content of his *total collapse theorem*, namely: if the system's total angular momentum $\mathbf{c}$ (a measure of the its global rotation) is non-zero, then collapse cannot happen (equivalently, if collapse occurs, then necessarily $\mathbf{c} = \mathbf{0}$). [26]

The proof, valid for the general N-body problem, is based on two relatively simple results of intrinsic interest: the *Lagrange-Jacobi's identity* and *Sundman's inequality.* Both are expressed in terms of the system's *moment of inertia*:

$$I \equiv \frac{1}{2} \sum_{j=1}^{N} m_j \mathbf{r}_j^2, \tag{4}$$

which is a measure of the spatial distribution of the masses. [27] As we are using the center-of-mass frame, a simple algebra shows that:

$$I = \frac{1}{2M} \sum_{j<k}^{N} m_j m_k r_{jk}^2, \tag{5}$$

where M is the total mass of the system. For future use, we note that Eq. (5) implies that if total collapse happens at time t^*, then $I(t^*) = 0$; thus, from Eq (4), it follows that the system collapses at the origin.

Lemma (Lagrange-Jacobi's identity)**.** *Let h be the total energy of the system, then:*

$$\ddot{I} = 2T - U = T + h = U + 2h. \tag{6}$$

Proof: The last two equalities in (6) follow from energy conservation. As for the first one, we just carry out the two derivatives using the chain-rule. We have:

$$\dot{I} = \sum_{j=1}^{N} m_j \mathbf{r}_j \cdot \mathbf{v}_j \Rightarrow \ddot{I} = \sum_{j=1}^{N} m_j \mathbf{v}_j^2 + \sum_{j=1}^{N} m_j \mathbf{r}_j \cdot \ddot{\mathbf{r}}_j = 2T + \sum_{j=1}^{N} \mathbf{r}_j \cdot \nabla_{\mathbf{r}_j} U, \tag{7}$$

where the last equality follows by plugging in Newton's equation (1). Now, the potential energy is a homogeneous function of degree -1, that is: $U(\lambda \mathbf{x}) = \lambda^{-1} U(\mathbf{x})$. Applying Euler's theorem of advanced calculus [28] we then get:

$$-U(\mathbf{x}) = \sum_{j=1}^{N} \mathbf{r}_j \cdot \nabla_{\mathbf{r}_j} U(\mathbf{x}), \tag{8}$$

which, when substituted in Eq. (7), gives the first identity in (6).

Lemma (Sundman's inequality). *Consider the system's total angular momentum* $\mathbf{c} = \sum_{j=1}^{N} m_j \mathbf{r}_j \wedge \mathbf{v}_j$ *and let* $c = \|\mathbf{c}\|$. *Then,*

$$c^2 \leq 4I(\ddot{I} - h). \tag{9}$$

Proof: This is a nice application of the Cauchy-Schwarz inequality from linear algebra:

$$c \leq \sum_{j=1}^{N} m_j \|\mathbf{r}_j \wedge \mathbf{v}_j\| \leq \sum_{j=1}^{N} m_j \|\mathbf{r}_j\| \|\mathbf{v}_j\| = \sum_{j=1}^{N} (\sqrt{m_j}\|\mathbf{r}_j\|)(\sqrt{m_j}\|\mathbf{v}_j\|) \tag{10}$$

$$\leq \sqrt{\sum_{j=1}^{N} m_j \mathbf{r}_j^2} \sqrt{\sum_{j=1}^{N} m_j \mathbf{v}_j^2} = \sqrt{2I}\sqrt{2T} = \sqrt{4IT}. \tag{11}$$

In sum, we have:

$$c^2 \leq 4IT, \tag{12}$$

from which we get (9) by using Lagrange-Jacobi's identity, (6).

We can at last state and prove Sundman's theorem (already known to Weierstrass for the three-body problem).

Theorem 1 (The total collapse theorem). *If total collapse happens, then* $\mathbf{c} = \mathbf{0}$. [29]

Proof: Let t^* be the time of total collapse, assumed positive without loss of generality. As a collapse is a multiple collision, we have:

$$\lim_{t \to t^*} U(t) = +\infty. \tag{13}$$

Therefore, by Lagrange-Jacobi's identity we get:

$$\lim_{t \to t^*} \ddot{I}(t) = +\infty. \tag{14}$$

From (14) it follows that for all t in a neighborhood of t^*, with $t < t^*$, we have $\ddot{I}(t) > 0$. As $I(t) > 0$ and recalling that $I(t^*) = 0$, it follows from calculus that $I(t)$

is strictly decreasing function in this neighborhood. Hence, for $t_1 \leq t \leq t_2$, with $t_2 < t^*$, we have $-\dot{I}(t) > 0$.

Now, consider Sundman's inequality in the form:

$$\ddot{I}(t) \geq \frac{c^2}{4I(t)} + h, \tag{15}$$

for $t_1 \leq t \leq t_2$. Multiplying (15) by $-\dot{I}(t) > 0$, we get:

$$-\dot{I}\ddot{I} \geq -\frac{c^2}{4}\frac{\dot{I}}{I} - h\dot{I}, \tag{16}$$

or better still:

$$-\frac{1}{2}\frac{d}{dt}(\dot{I})^2 \geq -\frac{c^2}{4}\frac{d}{dt}\ln(I) - h\frac{d}{dt}I. \tag{17}$$

Integrating both sides of (17) from t_1 to t_2, we get:

$$-\frac{1}{2}[\dot{I}^2(t_2) - \dot{I}^2(t_1)] \geq \frac{c^2}{4}\ln[I(t_1)/I(t_2)] - h[I(t_2) - I(t_1)], \tag{18}$$

or, regrouping,

$$\frac{c^2}{4}\ln[I(t_1)/I(t_2)] \leq h[I(t_2) - I(t_1)] + \frac{1}{2}[\dot{I}^2(t_1) - \dot{I}^2(t_2)]. \tag{19}$$

But, $I(t_2) - I(t_1) \leq I(t_2)$ and $\dot{I}^2(t_1) - \dot{I}^2(t_2) \leq \dot{I}^2(t_1)$, so from (19) we get:

$$\frac{c^2}{4} \leq \frac{hI(t_2) + \dot{I}^2(t_1)}{\ln[I(t_1)/I(t_2)]}. \tag{20}$$

Finally, observe that the right-hand side of (20) goes to zero as t_2 tends to t^* (for each fixed t_1). As c is constant, we conclude that $c = 0$, which completes the proof.

From Sundman's theorem it follows that any initial conditions leading to collapse must belong to the set defined by the equations $\mathbf{c} = \mathbf{0}$. But these equations specify a lower dimensional hyper-surface in phase-space, having therefore zero volume (much as a circle has zero area in the plane). *If* we then agree that subsets of zero volume are "rare" or "atypical" (and in a sense, negligible), one could say that "typical" (or "practically all") solutions of the N-body problem are collapse-free, which is a bit reassuring.

V Aftermath and conclusions

In 1912 Sundman succeeded in finding the general exact solution of the three-body problem as an infinite series in powers of $t^{1/3}$, for all time t, and for all initial

conditions (except for the negligible set leading to ternary collisions). [30] He thus cracked a problem that had resisted the attempts of the greatest mathematical-physicists of all times; so, even if a little too late, the Oscar definitely should go to Sundman.

But there is an ironic twist to the story. It soon became clear that Sundman's solution gave no new information about the general behavior of the system. Going from bad to worse, it was later shown that it is also useless for numerical calculations due to the incredibly slow rate of convergence of the series: it is estimated that of the order of $10^{8.000.000}$ terms would be needed to match the precision of astronomical measurements! [31] Hence, though justly hailed as an important conceptual achievement, his solution didn't lay bare the secrets of the three-body problem, probably explaining why it is hardly mentioned in the physics literature.

Somewhat paradoxically, Sundman's solution shows that finding an exact solution does not always improve our understanding of a problem. The positive side of all that (and of the previous impossibility results) was that it ultimately led to a momentous change from a predominantly *quantitative* to a new *qualitative* approach in dynamics, a culmination of a long historical process. [32] Pioneered by Poincaré, Lyapunov and others, it realizes that in dealing with such complicated dynamical systems as the N-body problem, the focus should move from finding particular solutions to a study of *families* of them (and even of families of systems!). One should try to figure out, as an expert said, "most of the dynamics of most systems", [33] that is, the *typical* behavior and properties (for instance, but not exclusively, those valid except for a "small" set of initial conditions). Elsewhere, we argued that a similar change was taking place, more or less at the same time, in Ludwig Boltzmann's statistical approach to kinetic gas theory. [34]

Another interesting aspect of this story is the emergence of the singularities. In spite of appearing in many branches of physics (e.g., in statistical mechanics, hydrodynamics, general relativity) it is surprising the lack of attention devoted to a unified understanding of their role and meaning. An investigation could bring some additional understanding to the delicate question of mathematical idealizations in physical models. There are many interesting issues to examine: is a singularity a signal of the breakdown of physical laws; is it a mathematical way to describe some underlying peculiar ("non-smooth") phenomenon; or is it just an artifact arising from a simplified model? In the case of the N-body problem it could be argued that the last case applies, as the use of point masses, with their infinite source of potential energy, is physically untenable. However, this model is pretty robust in capturing some essential features of real systems. In this sense it would be nice to have at least a proof that singularities are rare. For the moment, it is known that the set of initial data leading to collisions of any kind has zero volume, for all N. As for singularities of any kind, the same is true in the $N = 4$ case, while it is an open problem for $N \geq 5$. In other words, one doesn't even know whether or not the general N-body problem has global solutions for "most" initial conditions!

It is said that Newton complained of a headache when he tackled the three-body

problem in the guise of the Sun-Earth-Moon system. Today, with more than 300 years of hindsight and despite the great advances in non-linear dynamics, numerical analysis and computer simulations, even the experts admit that "the three-body problem is as enigmatic as ever". [35] The headache continues and it seems it will persist for quite some time.

Notes

[1]See, for instance, C. M. Linton, *From Eudoxus to Einstein. A History of Mathematical Astronomy* (Cambridge University Press, NY, 2004) and also M. Gutzwiller, "Moon-Earth-Sun: the oldest three-body problem", Reviews of Modern Physics, **70** (2), 589-639 (1998).

[2]V. I. Arnold, *Huygens & Barrow, Newton & Hooke* (Birkhäuser, Basel, 1990), Ch. 1.

[3]See D. L. Goroff, "Henri Poincaré and the birth of chaos theory: an introduction to the english translation of *Les Méthodes Nouvelles de la Mécanique Céleste*" in *New Methods of Celestial Mechanics (History of Modern Physics)*, H. Poincaré and D. L. Goroff (Ed.), Vol. 1 (AIP Press, 1992), p. I17.

[4]For a review, see J. E. Marsden and S. D. Ross, "New methods in celestial mechanics and mission design", Bull. AMS, **43**, 43-73 (2006)

[5]See, respectively, D. L. Smith, "Next Exit 0.5 Million Kilometers", Engineering & Science, **LXV** (4), 6-15, (2002) and R. Montgomery, "A New Solution to the Three-Body Problem", Notices of the AMS, **48** (5), 471-481 (2001).

[6]From N. Grossman, *The Sheer Joy of Celestial Mechanics* (Birkhäuser, Boston, 1996).

[7]Additional credibility to the model came from Newton's proof that the gravitational force outside a spherically symmetric body is the same as if all mass were concentrated in its center. However, the oblate shape is more realistic and can have important effects.

[8]Observe the exclusion of configurations belonging to the *singular set* $\Delta = \cup_{1\leq i<j\leq N}\Delta_{ij}$, where $\Delta_{ij} = \{\mathbf{x} = (\mathbf{r}_1, \cdots, \mathbf{r}_N) \in \mathbb{R}^{3N} : \mathbf{r}_i = \mathbf{r}_j\}$.

[9]M. Bunge, *Chasing Reality. Strife over Realism* (University of Toronto Press, Toronto, 2006), pp. 151.

[10]For example, an anecdote tells of a lecture delivered by mathematician Mark Kac at Caltech, to which Feynman attended. After the lecture Feynman got up and announced: "If all mathematics disappeared, it would set physics back by precisely one week." To which Kac immediately replied: "Precisely the week in which God created the world."

[11]By the way, such a theorem, first due to Cauchy, was distilled from some numerical methods used in practice by astronomers; see A. Wintner, *Analytical Foundations of Celestial Mechanics* (Princeton University Press, 1964), p. 143.

[12]As usual, we take the center-of-mass reference frame so that $\mathbf{R}_{cm} = \mathbf{P} = \mathbf{0}$.

[13]They were known to Lagrange as of 1772, in the context of the three-body problem, who also obtained an additional reduction by "elimination of time".

[14]Jacobi later discovered an additional "reduction of the nodes".

[15]See ref. 11.

[16]See J. Laskar, "La stabilité du système solaire", in *Chaos et déterminisme*, edited by A. D. Dalmedico, J. -L. Chabert and K. Chemla (Éditions du Seuil, Paris, 1992), pp. 171-211.

[17]See V. I. Arnold, V. V. Kozlov and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics* (Springer, Berlin, 1997).

[18]Ref. 1, pp. 85.

[19]The proof contained some mistakes. It was later corrected and extended by Poincaré and Painlevé, but still contained some errors which were cleared only recently in E. Julliard-Tosel, "Brun's Theorem: the Proof and Some Generalizations", Celest. Mechs., **76**, 241-281 (2000).

[20]Quoted from J. Barrow-Green, *Poincaré and the Three-Body Problem* (American Mathematical Society, 1997), pp. 229.

[21]For details, see D. G. Saari, *Collisions, Rings and Other Newtonian N-Body Problems* (American Mathematical Society, CBMS, Regional Conference Series in Mathematics, Number 104, Rhode Island, 2005).

[22]Though this is no contradiction with the model at hand, it is still remarkable that Newton's equations allow such bizarre solutions. In the known examples, use is made of the inexhaustible source of energy from Newtonian point-mass potential, see D. G. Saari and Z. Xia, "Off to Infinity in Finite Time", Notices of the AMS, **42** (5), 538-546 (1995).

[23]This is known as *Painlevé's Conjecture*, that was proved only recently, for $N \geq$

5; the case $N = 4$ it is still open. See F. Diacu, "Painlevé's Conjecture", The Mathematical Intelligencer, **15** (2), 6-12 (1993).

[24]For a discussion, see D. G. Saari, "A Visit to the Newtonian N-body Problem via Elementary Complex Variables", Am. Math. Monthly, **97** Feb., 105-119 (1990).

[25]Incidentally, the same happens for ternary collisions of hard "billiard balls": in contrast to binary collisions, the conservation laws are not enough to unambiguously fix the posterior motion.

[26]The converse is not true, except for the two-body problem. Examples are motions confined to a line, but then binary collisions necessarily occur in the future or the past. Another (collision less) example is the recently discovered figure *eight* periodic solution of the three-body problem.

[27]We follow the tradition in celestial mechanics of putting a 1/2 factor.

[28]See J. E. Marsden and A. J. Tromba, *Vector Calculus* (W H Freeman and Company, San Francisco, 1981), 2nd. ed., pp. 148, ex. 21.

[29]Sundman proved a stronger result: if $\mathbf{c} \neq \mathbf{0}$, then o $\max_{i\neq k} r_{ik} \geq D(\mathbf{c}) > 0$, so that the particles remain strictly isolated from triple collisions.

[30]For a discussion, see M. Henkel, "Sur la solution de Sundman du problème des trois corps", Philosophia Scientiae, **5** (2), 161-184 (2001) and for details of the proof see C. Siegel and J. Moser, *Lectures in Celestial Mechanics*, (Springer-Verlag, NY, 1971). The generalization for $N \geq 4$ was obtained much later, see Q. Wang, "The global solution of the N-body problem", Celest. Mechs, **50**, 73-88 (1991).

[31]D. Belorizky, "Recherches sur l'application pratique des solutions générales du probléme des trois corps", Journal des Observateurs, **16** (7), 109-131 (1933).

[32]See M. W. Hirsch, "The Dynamical System's Approach to Differential Equations", Bulltin of the AMS, **11** (1), (1984), section 6, and also A. Chenciner , De la Mécanique céleste á la théorie des systémes dynamiques, aller et retour, in actes de la confrence "Epistmologie des systmes dynamiques", Paris Dcembre 1999, to appear.

[33]J.-C. Yoccoz, "Recent developments in dynamics", in *Proceedings of the International Congress of Mathematicians*, **1** (Birkhäuser, Basel, 1995) pp. 247-265.

[34]See S. B. Volchan, "Probability as typicality", Studies in History and Philosophy of Modern Physics, **38** (4), 801-814 (2007).

[35]C. D. Murray and S. F. Dermott, *Solar System Dynamics* (Cambridge University Press, NY, 2005), p. 63.

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